



NORTH-HOLLAND

Sequence Operators From Groups

Peter J. Cameron

School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
London E1 4NS
U.K.

Dedicated to J. J. Seidel

Submitted by W. H. Haemers

ABSTRACT

This paper is inspired by the paper *Some Canonical Sequences of Integers* by Bernstein and Sloane. The main observation is that seven of the operators defined in that paper have natural interpretations in terms of counting orbits of groups, providing a pattern which is completed by five further operators. Some of their eigen-sequences also have group-theoretic meaning.

1. INTRODUCTION

A permutation group G on an infinite set Ω is *oligomorphic* if, for each positive integer n , there are only finitely many G -orbits on Ω^n . These groups have close connections with logic: the first-order theory of a countable structure is \aleph_0 -categorical if and only if $\text{Aut}(M)$ is oligomorphic (the Engeler-Ryll-Nardzewski-Svenonius theorem). There are also strong links with enumerative combinatorics (see below). More information can be found in Cameron [3].

Associated with an oligomorphic permutation group, there are three obvious sequences of integers:

- $F_n^*(G)$, the number of G -orbits on Ω^n ;
- $F^n(G)$, the number of G -orbits on ordered n -tuples of distinct points;
- $f_n(G)$, the number of G -orbits on n -sets.

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The first two sequences carry equivalent information since

$$F_n^*(G) = \sum_{k=1}^n S(n, k) F_k(G),$$

that is, the first is the Stirling transform of the second (in the terminology of [1]). Sometimes the first is more natural, e.g., $F_n^*(\text{Aut}(M))$ is the number of n -types in the \aleph_0 -categorical theory of the countable structure M . But I will consider only the second and third sequences below. When I speak of generating functions, I will always mean the ordinary generating function $f_G(x) = \sum f_n(G)x^n$ of the third sequence, and the exponential generating function $F_G(x) = \sum F_n(G)x^n/n!$ of the second. (By convention, $f_0 = F_0 = 1$.)

Now suppose that M is a countable structure such that $G = \text{Aut}(M)$ is oligomorphic. Without loss of generality, we may suppose that M is *homogeneous*, that is, any isomorphism between finite substructures of M extends to an automorphism of M . The *age* of M , $\text{Age}(M)$, is the class of finite substructures of M . Now

- $f_n(G)$ is the number of unlabelled n -element structures in $\text{Age}(M)$;
- $F_n(G)$ is the number of labelled n -element structures in $\text{Age}(M)$.

A theorem of Fraïssé gives a necessary and sufficient condition for a class of finite structures to be the age of a countable homogeneous structure. For any class satisfying this condition (of which many exist!), the unlabelled and labelled enumeration problems are thus equivalent to calculating $f_n(G)$ and $F_n(G)$, respectively, for an appropriate oligomorphic group G .

In fact, for any oligomorphic group G , there is a power series in infinitely many variables, the *modified cycle index*, of which both $f_G(x)$ and $F_G(x)$ are specializations. The series is itself a special case of Joyal's cycle index of a species [4]. (Joyal would presumably regard an age as a species.)

2. OPERATORS FROM GROUPS

Let H and K be permutation groups on sets Γ and Δ , respectively. Then the direct product $H \times K$ is a permutation group on the disjoint union $\Gamma \cup \Delta$ and the wreath product $H \text{ Wr } K$ is a permutation group on the Cartesian product $\Gamma \times \Delta$ (regarded as a cover of Δ with fibers isomorphic to Γ). The point stabilizer H_α in a transitive group H acts on the points different from α .

These group-theoretic constructions can be used to define operators on sequences. Once we have understood how these operators work, they can be

extended to all sequences (a_n) of positive integers with $a_0 = 1$. In terms of generating functions, we have

$$\begin{aligned} F_{H \times K}(x) &= F_H(x) \cdot F_K(x), \\ f_{H \times K}(x) &= f_H(x) \cdot f_K(x), \\ F_{H \text{ Wr } K}(x) &= F_K(F_H(x) - 1). \end{aligned}$$

The sequence $(f_n(H \text{ Wr } K))$ is not uniquely determined by the sequences $(f_n(H))$ and $(f_n(K))$; we require $(f_n(H))$ and the modified cycle index of K . This is the most interesting case! Also, $F_n(H_\alpha) = F_{n+1}(H)$ if H is transitive; so $F_H(x) = F'_H(x)$. (The sequence $(f_n(H_\alpha))$ is not uniquely determined by $(f_n(H))$; it requires knowledge of the modified cycle index of H .)

Two particular groups we require are:

- S , the symmetric group on a countable set;
- A , the group of order-preserving permutations of the rationals.

We have $F_S(x) = e^x$, $F_A(x) = f_S(x) = f_A(x) = 1/(1 - x)$.

Now we consider some operators. Where these occur in [1], I have used the same names. I have also used **SUM** for the operator which maps a sequence to its sequence of partial sums. The other new operators are defined below. The columns give, respectively, the map of groups and the effect on the sequences (f_n) and (F_n) . The first row requires that G is transitive. L is the left shift.

Group	Unlabelled	Labelled
$G \rightarrow G_\alpha$	undefined	L
$G \rightarrow G \times G$	CONV	EXP-CONV
$G \rightarrow G \text{ Wr } G$	undefined	O1
$G \rightarrow G \times S$	SUM	BINOMIAL
$G \rightarrow G \times A$	SUM	O2
$G \rightarrow G \text{ Wr } S$	EULER	EXP
$G \rightarrow G \text{ Wr } A$	INVERT	O3
$G \rightarrow S \text{ Wr } G$	undefined	STIRLING
$G \rightarrow A \text{ Wr } G$	undefined	O4

The operators **O1–O4** are defined by their effect on e.g.f.'s as follows. As in [1], $A(x)$ and $\mathcal{A}(x)$ denote the ordinary and exponential generating

functions of a sequence (a_n) , and $B(x)$ and $\mathcal{B}(x)$ those of its transform by the operator under consideration:

$$\mathbf{O1}: \mathcal{B}(x) = \mathcal{A}(\mathcal{A}(x) - 1);$$

$$\mathbf{O2}: \mathcal{B}(x) = \mathcal{A}(x)/(1 - x);$$

$$\mathbf{O3}: \mathcal{B}(x) = 1/(2 - \mathcal{A}(x));$$

$$\mathbf{O4}: \mathcal{B}(x) = \mathcal{A}(x)/(1 - x).$$

Other groups define further operators. For example, let C be the group preserving the cyclic order of the complex roots of unity. We have $f_C(x) = 1/(1 - x)$ and $F_C(x) = 1 - \log(1 - x)$; this determines all the operators except the entry under “Unlabelled” corresponding to $G \mapsto G \text{ Wr } C$, which is defined on the ordinary generating function by

$$B(x) = 1 - \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log(2 - A(x^n)).$$

where ϕ is Euler’s totient, and the entry under “Unlabelled” for $G \mapsto C \text{ Wr } G$, which is undefined.

3. EIGEN-SEQUENCES

It seems to happen quite often that eigen-sequences of these operators are associated with interesting groups. In most cases, there is an obvious reason; but a few can be explained by structural properties of various groups.

Suppose that Φ is some operator on groups, and \mathbf{O} the operator on sequences corresponding to $(F_n(G)) \mapsto (F_n(\Phi(G)))$. If there exists a group G which satisfies the “differential equation” $G_\alpha \cong \Phi(G)$, then the sequence $(F_n(G))$ is shifted left by the operator \mathbf{O} .

For example, the group $G = A$ satisfies $A_\alpha \cong A \times A$ (the two factors acting on the left and the right of α). So **EXP-CONV** $\circ F(A) = L \circ F(A)$, in agreement with a property of $F_n(A) = n!$ noted in [1] (sequence S15 in the table). We see that the same equation shows that the factorials are also shifted left by the operator **O2** defined in the preceding section.

Similarly, $G = S \text{ Wr } S$ satisfies $G_\alpha \cong G \times S$, so **BINOMIAL** $\circ F(G) = L \circ F(G)$. Now $F_n(G) = F_n^*(S)$ is the n th Bell number (sequence S1 in [1]) giving a known characterization of multiple transitivity: a group G is n -transitive if and only if $F_n^*(G)$ is the n th Bell number. (G is n -transitive if and only if $F_m(G) = 1$ for $m \leq n$.)

There is a group G satisfying $G_\alpha \cong G \text{ Wr } A$, defined in Cameron [2] (and called ∂T_3 there); it is the automorphism group of a ternary relational structure associated with the leaves of a binary tree. The sequence $F(G)$,

where $F_1(G) = 1$ and $F_n(G) = (2n - 3)!! = 1 \cdot 3 \cdot 5 \cdots (2n - 3)$ for $n > 1$, is shifted left by the operator **O3**.

It can be shown that no transitive permutation group G can satisfy $G_\alpha \cong G \text{ Wr } S$. However, the group C of the preceding section does satisfy $F_n(C_\alpha) = F_n(C \text{ Wr } S)$ for all n , corresponding to the decomposition of permutations into disjoint cycles. So **EXP** $\circ F(C) = L \circ F(C)$ (see sequence S15 in [1]).

The sequences shifted left by **O1** and **O4**, commencing

$$1, 1, 2, 7, 37, 269, 2535, \dots$$

and

$$1, 1, 3, 15, 111, 1131, 15081, \dots$$

respectively, appear to be new. I know no groups satisfying the corresponding “differential equations” $G_\alpha \cong G \text{ Wr } G$ or $G_\alpha \cong A \text{ Wr } G$.

Similar group-theoretic facts sometimes translate into identities between the operators. For example, for any transitive group G , we have

$$(S \text{ Wr } G)_\alpha \cong S \times (S \text{ Wr } G_\alpha),$$

whence

$$L \circ \mathbf{STIRLING} = \mathbf{BINOMIAL} \circ \mathbf{STIRLING} \circ L.$$

This identity translates into the familiar relation

$$\sum_{l=k}^n \binom{n-1}{l} S_{l,k} = S_{n,k+1}.$$

It also shows that **STIRLING** maps the all-one sequence (the eigen-sequence of L) to the Bell numbers S1 (the solution of **BINOMIAL** $\circ a = L \circ a$).

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